Solution of HW5

Compulsory Part:

10 Since X_0 has the stationary distribution π, X_1 also has the stationary distribution π. Note that

$$P(X_0 = y \mid X_1 = x) = \frac{P(X_0 = y, X_1 = x)}{P(X_1 = x)} = \frac{\pi(y)P(y,x)}{\pi(x)}.$$

It suffices to show that for any $x, y \in \mathcal{S}$, $\pi(x)P(x,y) = \pi(y)P(y,x)$. For $y = x$ or $|y - x| \geq 2$, the equation is trivial. If $y = x+1$, then by (9),

$$\pi(x)P(x,x+1) = \pi(0)\pi_x p_x = \pi(0)\frac{p_0 \cdots p_{x-1}p_x}{q_1 \cdots q_x} = \pi(0)\pi_{x+1}q_{x+1} = \pi(x+1)P(x+1,x).$$

If $y = x - 1$, $x \geq 1$, then by (9),

$$\pi(x)P(x,x-1) = \pi(0)\pi_x q_x = \pi(0)\frac{p_0 \cdots p_{x-1}}{q_1 \cdots q_{x-1}} = \pi(0)\pi_{x-1}p_{x-1} = \pi(x-1)P(x-1,x).$$

15 Let $\mathcal{S} = \{1, 2, \ldots, d\}$ be the state space. Since all states are in a finite irreducible closed set, they are positive recurrent. Thus the stationary distribution is unique (page 68, Corollary 7).

Let $\pi(x) = \frac{1}{d}$ for all $x \in \mathcal{S}$. Then it is a probability vector since $\sum_{x=1}^{d} \pi(x) = 1$. Moreover, for all $y \in \mathcal{S}$,

$$\sum_{x=1}^{d} \pi(x)P(x,y) = \sum_{x=1}^{d} \frac{1}{d}P(x,y) = \frac{1}{d} = \pi(y).$$

This shows π is the unique stationary distribution we want.

20 **(a)** For the irreducible closed set $\{0, 1\}$, its transition matrix is given by $P_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$. Let $\pi_1 = (\pi_1(0), \pi_1(1))$ and then solve

$$\begin{cases} \pi_1 P_1 = \pi_1, \\ \pi_1(0) + \pi_1(1) = 1. \end{cases}$$

We get $\pi_1 = (\frac{2}{5}, \frac{3}{5})$. Hence the stationary distribution concentrated on $\{0, 1\}$ is given by $(\frac{2}{5}, \frac{3}{5}, 0, 0, 0, 0)$.

For the irreducible closed set $\{2, 4\}$, its transition matrix is given by $P_2 = \begin{bmatrix} \frac{1}{8} & \frac{7}{8} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}$. Let $\pi_2 = (\pi_2(2), \pi_2(4))$ and then solve

$$\begin{cases} \pi_2 P_2 = \pi_2, \\ \pi_2(2) + \pi_2(4) = 1. \end{cases}$$

We get $\pi_2 = (\frac{6}{13}, \frac{7}{13})$. Hence the stationary distribution concentrated on $\{2,4\}$ is given by $(0, 0, \frac{6}{13}, 0, \frac{7}{13}, 0)$.

(b) We use Theorem 1 in textbook, page 58. If y is recurrent and $\pi(y)$ is the stationary distribution concentrared on the corresponding irreducible closed set,

$$\lim_{n\to\infty} \frac{G_n(x,y)}{n} = \frac{\rho_{xy}}{m_y} = \rho_{xy} \cdot \pi(y).$$

If y is transient, it is clear that $\lim_{n\to\infty} \frac{G_n(x,y)}{n} = 0$. As all ρ_{xy} and $\pi(y)$ are computed before, we have

$$[\lim_{n\to\infty} \frac{G_n(x,y)}{n}]_{0\le x,y\le 5} = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 0 & 0 & 0 & 0 \\ \frac{2}{5} & \frac{3}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{6}{13} & 0 & \frac{7}{13} & 0 \\ \frac{14}{55} & \frac{21}{55} & \frac{24}{143} & 0 & \frac{28}{143} & 0 \\ 0 & 0 & \frac{6}{13} & 0 & \frac{7}{13} & 0 \\ \frac{12}{55} & \frac{18}{55} & \frac{30}{143} & 0 & \frac{35}{143} & 0 \end{bmatrix}.$$

21 The stationary distribution is given by Q7(a):

$$\pi = (\pi(0), \pi(1), \pi(2), \pi(3), \pi(4)) = (\frac{1}{16}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{16}).$$

The period of the chain is 2.

(a) It follows from Theorem 7 in page 73 that for n large and even

$$\big(P_0(X_n = x)\big)_{0\le x\le 4} = \big(P^n(0,x)\big)_{0\le x\le 4} \doteq (2\pi(0), 0, 2\pi(2), 0, 2\pi(4)) = (\frac{1}{8}, 0, \frac{3}{4}, 0, \frac{1}{8}).$$

(b) It follows from Theorem 7 in page 73 that for n large and odd

$$\big(P_0(X_n = x)\big)_{0\le x\le 4} = \big(P^n(0,x)\big)_{0\le x\le 4} \doteq (0, 2\pi(1), 0, 2\pi(3), 0,) = (0, \frac{1}{2}, 0, \frac{1}{2}, 0).$$

22 **(a)** Denote $i \to j$ if $P(i,j) > 0$, where P is the transition probability. Note that in this matrix

$$0 \to 2 \to 1 \to 0,$$

the chain is irreducible.

(b) Note that

$$P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}; \quad P^3 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix},$$

thus $P^2(0,0) > 0$ and $P^3(0,0) > 0$, the period of 0 is given by $d_0 = g.c.d.\{n : P^n(0,0) > 0\} = 1$.

(c) Let π be the stationary distribution. Then $\pi(0) + \pi(1) + \pi(2) = 1$. Solve the equation $\pi P = \pi$. We have $\pi = (\frac{2}{5}, \frac{1}{5}, \frac{2}{5})$.

23 **(a)** Since

$$0 \to 1 \to 3 \to 0 \to 2 \to 4 \to 0,$$

the chain is irreducible.

(b) Since $P(0,0) = 0$, $P^2(0,0) = 0$, $P^3(0,0) \geq P(0,1)P(1,3)P(3,0) > 0$, together with $P^4 = P$, the period of the chain is 3.

(c) Let π be the stationary distribution. Then $\pi(0)+\pi(1)+\pi(2)+\pi(3)+\pi(4) = 1$. Solve the equation $\pi P = \pi$. We have $\pi = (\frac{1}{3}, \frac{1}{9}, \frac{2}{9}, \frac{1}{12}, \frac{1}{4})$.

Optional Part:

11 We use induction on n. For $n = 0$, X_0 has a Poisson distribution with parameter $t = tp^0 + \frac{\lambda}{q}(1-p^0)$. Suppose that X_n has a Poisson distribution with parameter $tp^n + \frac{\lambda}{q}(1-p^n)$ for some $n \geq 0$. Then applying the result in page 54 of the textbook, $R(X_n)$ has a Poisson distribution with parameter $p(tp^n + \frac{\lambda}{q}(1-p^n)) = tp^{n+1} + \frac{\lambda}{q}(1-p^{n+1}) - \lambda$. Set $\mu_n = tp^{n+1} + \frac{\lambda}{q}(1-p^{n+1}) - \lambda$. Then for $x \geq 0$,

$$\begin{aligned}
P(X_{n+1} = x) &= P(\xi_{n+1} + R(X_n)) \\
&= \sum_{y=0}^{x} P(R(X_n) = y, \xi_{n+1} = x - y) \\
&= \sum_{y=0}^{x} P(R(X_n) = y) P(\xi_{n+1} = x - y) \\
&= \sum_{y=0}^{x} \frac{\mu_n^y e^{-y}}{y!} \frac{\lambda^{x-y} e^{-(x-y)}}{(x-y)!} \\
&= \frac{e^{-x}}{x!} \sum_{y=0}^{x} \binom{x}{y} \mu_n^y \lambda^{x-y} \\
&= \frac{(\mu_n + \lambda)^x e^{-x}}{x!}
\end{aligned}$$

which shows that X_{n+1} has the Poisson distribution with parameter $\mu_n + \lambda = tp^{n+1} + \frac{\lambda}{q}(1-p^{n+1})$. By induction, X_n has the indicated Poisson distribution.

16 For any $x \in \mathcal{S}$,

$$\sum_{y \in \mathcal{S}} Q(x,y) = 1 - p_x + \sum_{y \in \mathcal{S}: y \neq x} p_x P(x,y) = 1 - p_x + p_x \sum_{y \in \mathcal{S}: y \neq x} P(x,y) = 1 - p_x + p_x = 1.$$

Hence Q is the transition function of a Markov chain.

For $x, y \in \mathcal{S}$, since x leads to y in the Markov chain with respect to the transition function P, there exists a positive integer n, and $x_1, x_2, \cdots x_{n-1} \in \mathcal{S}$ such that

$$P(x, x_1)P(x_1, x_2) \cdots P(x_{n-1}, y) > 0.$$

This implies

$$Q(x,x_1)Q(x_1,x_2)\cdots Q(x_{n-1},y) = p_x p_{x_1}\cdots p_{x_{n-1}} P(x,x_1)P(x_1,x_2)\cdots P(x_{n-1},y) > 0.$$

Thus x leads to y in the Markov chain with respect to the transition function Q. Therefore the new chain is irreducible.

Since the state space $\mathcal{S}$ is finite, all states are positive recurrent, hence the new chain has a unique stationary distribution (page 68, Corollary 7).

Let $\pi'(x) = \frac{p_x^{-1}\pi(x)}{\sum_{y\in\mathcal{S}} p_y^{-1}\pi(y)}$, $x \in \mathcal{S}$. Then clearly $\pi'(x) \geq 0$,

$$\sum_{x\in\mathcal{S}} \pi'(x) = \frac{\sum_{x\in\mathcal{S}} p_x^{-1}\pi(x)}{\sum_{y\in\mathcal{S}} p_y^{-1}\pi(y)} = 1,$$

and for any $z \in \mathcal{S}$,

$$\begin{aligned}
(\pi' Q)(z) &= \sum_{x\in\mathcal{S}} \pi'(x)Q(x,z) \\
&= \frac{\sum_{x\in\mathcal{S}:x\neq z} p_x^{-1}\pi(x)p_x P(x,z) + p_z^{-1}\pi(z)(1-p_z)}{\sum_{y\in\mathcal{S}} p_y^{-1}\pi(y)} \\
&= \frac{\sum_{x\in\mathcal{S}:x\neq z} \pi(x)P(x,z) - \pi(z) + p_z^{-1}\pi(z)}{\sum_{y\in\mathcal{S}} p_y^{-1}\pi(y)} \\
&= \frac{(\pi P)(z) - \pi(z) + p_z^{-1}\pi(z)}{\sum_{y\in\mathcal{S}} p_y^{-1}\pi(y)} \\
&= \frac{p_z^{-1}\pi(z)}{\sum_{y\in\mathcal{S}} p_y^{-1}\pi(y)} = \pi'(z).
\end{aligned}$$

Hence π' is the stationary distribution of the Markov chain with respect to the transition function Q.

17 Note that this chain is irreducible and positive recurrent and the stationary distribution is given by Q7(a):

$$\pi(n) = \frac{\binom{d}{n}}{2^d}, \quad 0 \leq n \leq d.$$

Hence the mean return time to state 0 is

$$m_0 = \frac{1}{\pi(0)} = 2^d$$

by Theorem 5 in page 64.

18 **(a)** Let $A = \{1,2,\ldots,c\}$ and $B = \{c+1, c+2, \ldots, c+d\}$.

For $x,y \in A$, $\rho_{xy} \geq P(x,c+1)P(c+1,y) = (1/d)(1/c) > 0$.

For $x,y \in B$, $\rho_{xy} \geq P(x,1)P(1,y) = (1/c)(1/d) > 0$.

For $x \in A$, $y \in B$, $\rho_{xy} \geq P(x,y) = 1/d > 0$ and $\rho_{yx} \geq P(y,x) > 0$.

Hence the chain is irreducible.

(b) Since the chain is irreducible and finite, it has a unique stationary distribution π.

For $y \in A$, we have

$$\pi(y) = (\pi P)(y) = \sum_{x\in B} \pi(x)P(x,y) = \frac{1}{c}\sum_{x\in B} \pi(x),$$

which implies

$$\sum_{y\in A} \pi(y) = \sum_{y\in A} \frac{1}{c} \sum_{x\in B} \pi(x) = \sum_{x\in B} \pi(x).$$

Note that $\sum_{x\in A\cup B} \pi(x) = 1$. Hence $\sum_{y\in A} \pi(y) = \sum_{x\in B} \pi(x) = 1/2$. Thus for any $y \in A$, $\pi(y) = \frac{1}{2c}$.

For $z \in B$, we have

$$\pi(z) = (\pi P)(z) = \sum_{x\in A} \pi(x)P(x,z) = \frac{1}{d}\sum_{x\in A} \pi(x) = \frac{1}{2d}.$$

Therefore, the stationary distribution is

$$\pi(x) = \begin{cases} \frac{1}{2c}, & x \in A, \\ \frac{1}{2d}, & x \in B. \end{cases}$$

19 **(a)** For the irreducible closed set $\{1,2,3\}$, its transition matrix is given by $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

This matrix is doubly stochastic. By Q15, the stationary distribution concentrated on $\{1,2,3\}$ is given by $(0, 1/3, 1/3, 1/3, 0, 0, 0)$.

For the irreducible closed set $\{4,5,6\}$, its transition matrix is given by $\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

This matrix is doubly stochastic. By Q15, the stationary distribution concentrated on $\{4,5,6\}$ is given by $(0, 0, 0, 0, 1/3, 1/3, 1/3)$.

(b) We use Theorem 1 in textbook, page 58. If y is recurrent and $\pi(y)$ is the stationary distribution concentrared on the corresponding irreducible closed set,

$$\lim_{n\to\infty} \frac{G_n(x,y)}{n} = \frac{\rho_{xy}}{m_y} = \rho_{xy} \cdot \pi(y).$$

If y is transient, it is clear that $\lim_{n\to\infty} \frac{G_n(x,y)}{n} = 0$. As all ρ_{xy} and $\pi(y)$ are computed before, we have

$$[\lim_{n\to\infty} \frac{G_n(x,y)}{n}]_{0\le x,y\le 6} = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$